An Interesting Problem on Hyperbolas

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1 Problem Statement

Problem. Consider the hyperbola given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Let P be any point on this hyperbola, and let the normal to the curve at P intersect the transverse axis AA' at M and the conjugate axis BB' at N. From the centre C of the hyperbola, drop a perpendicular to this normal, and denote its foot by T. Determine the value of the expression

$$\frac{PT \cdot PM}{CB^2},$$

where CB is the semi-conjugate axis of the hyperbola. (Figure 1)

2 Insights and Solution Sketch

We begin by assuming that the point P lies in the first quadrant; an entirely analogous argument can be constructed if P is located in any of the remaining three quadrants.

Before proceeding to the main argument, it is helpful to examine the geometric configuration in more detail.

> What is the position of the point T on the line MN? More precisely, is T located on the segment joining P and M, or on the segment joining P and N?

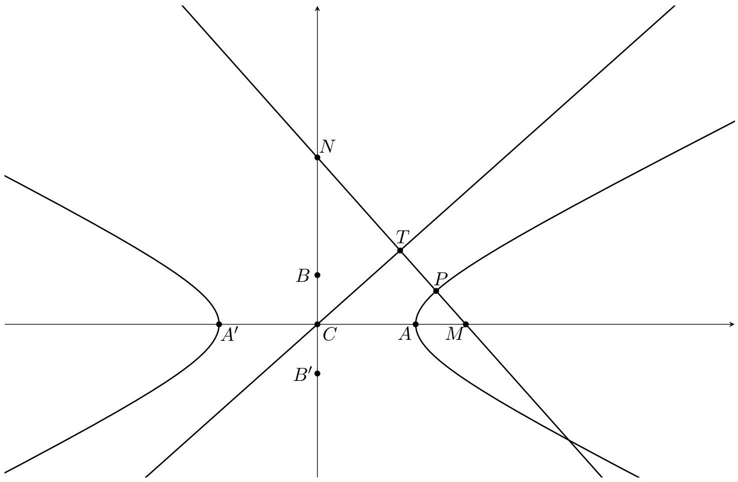

Figure 1: Determine the value of $\frac{PT \cdot PM}{CB^2}$

Observe that the line CT is parallel to the tangent at P to the given curve. This follows from the fact that both lines are perpendicular to the same line, the normal at P. Denote the coordinates of P by (x_1, y_1), with $x_1, y_1 > 0$ as P is assumed to lie in the first quadrant. The equation of the tangent to the hyperbola at the point P is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Setting $y = 0$ in the above equation, we find that it meets the x–axis at

$$x = \frac{a^2}{x_1},$$

which is strictly positive since $x_1 > 0$. Hence, the tangent at P intersects the positive x–axis.

Because CT is constructed to be parallel to this tangent, the line through C parallel to the tangent must intersect the normal at some point lying between P and N on the line MN. Consequently, the point T must lie on the segment PN.

We need to find the ratio $\frac{PT \cdot PM}{CB^2}$. We know that $CB^2 = b^2$. Thus, we can try using similar triangles to determine $PT \cdot PM$.

Let $X = (x_1, 0)$ denote the foot of the perpendicular from P to the x-axis. We observe that the triangles $\triangle TCM$ and $\triangle XPM$ are similar. This similarity allows us to write the corresponding ratios of sides:

$$\frac{TC}{XP} = \frac{CM}{PM} = \frac{TM}{XM}.$$

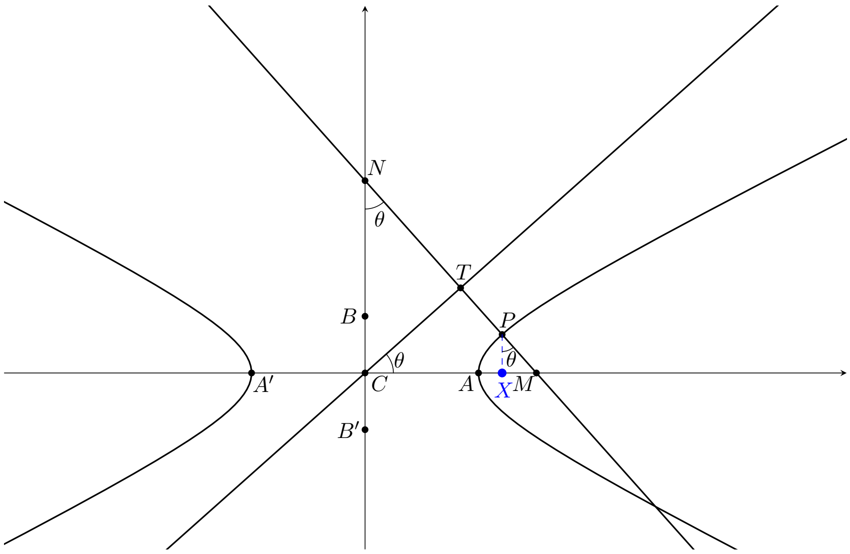

Figure 2: Point X is drawn such that $PX \perp x$-axis

3 Solution

Let X be the foot of the perpendicular from P to the x-axis, and let $P = (x_1, y_1)$. Observe that $\angle PXM = 90^\circ = \angle CTM$, and that $\angle CMT = \angle XPM$. Hence, the triangles $\triangle TCM$ and $\triangle XPM$ are similar. From this similarity, the corresponding sides are proportional:

$$\frac{CM}{PM} = \frac{TM}{XM}, \tag{1}$$

$$\implies CM \cdot XM = TM \cdot PM. \tag{2}$$

Since T lies between P and N on the line PN, we have $TM = PT + PM$. Substituting this into Eq. (2), we obtain

$$\begin{aligned} CM \cdot XM &= (PT + PM) \cdot PM \\ &= PT \cdot PM + PM^2, \\ \implies PT \cdot PM &= CM \cdot XM - PM^2. \end{aligned}$$

In the right-angled triangle $\triangle PXM$, the Pythagorean theorem gives

$$PM^2 = XP^2 + XM^2.$$

Therefore, we can rewrite the previous expression as

$$\begin{aligned} PT \cdot PM &= CM \cdot XM - XP^2 - XM^2 \\ &= XM(CM - XM) - XP^2 \\ &= XM \cdot XC - XP^2. \end{aligned}$$

Noting that $XP = y_1$ and $CX = x_1$, we have

$$PT \cdot PM = XM \cdot x_1 - y_1^2. \tag{3}$$

Next, we determine XM. The line CT is parallel to the tangent t to the hyperbola at P, whose equation is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

The slope of t is

$$m_t = \frac{b^2 x_1}{a^2 y_1},$$

and hence the slope of CT is the same. Denoting $\angle TCM = \theta$, we have

$$\tan\theta = \text{slope of } CT = \frac{b^2 x_1}{a^2 y_1}.$$

By similarity, $\angle TCM = \angle XPM = \theta$, so

$$\tan\theta = \frac{XM}{XP} = \frac{XM}{y_1}.$$

Comparing with the previous expression, we obtain

$$XM = \frac{b^2 x_1}{a^2}.$$

Substituting this into Eq. (3), we find

$$\begin{aligned} PT \cdot PM &= \frac{b^2 x_1}{a^2} \cdot x_1 - y_1^2 \\ &= \frac{b^2 x_1^2}{a^2} - y_1^2. \end{aligned}$$

Finally, dividing by $CB^2 = b^2$, the desired ratio is

$$\frac{PT \cdot PM}{CB^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$

Since $P = (x_1, y_1)$ lies on the hyperbola, we have $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$. Hence, the ratio evaluates to

$$\frac{PT \cdot PM}{CB^2} = \boxed{1}.$$